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MINIMUM CYCLE COVERINGS AND INTEGER FLOWS

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ABSTRACT

It was conjectured by Fan that if a graph $G=(V,E)$ has a nowhere-zero 3-flow, then G can be covered by two even subgraphs of total size at most $|V|+|E|-3$. This conjecture is proved in this paper. It is also proved in this paper that the optimum solution of the Chinese postman problem and the solution of minimum cycle covering problem are equivalent for any graph admitting a nowhere-zero 4-flow.

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We use notations of [BM]. Let $G=(V,E)$ be a graph with vertex set V and edge set E. An even subgraph of G is a subgraph of G such that the degree of each vertex is even in H. It is clear that an even subgraph is a union of edge-disjoint cycles. The set of all neighbors of a vertex v is denoted by $N(v)$.

Let D be an orientation of G, an assignment of a direction to each edge. Let f be a weight function on $E(G)$, an assignment of an integer $f(e)$ to each edge e. A k-flow of G is a pair (D, f), consisting of an orientation and a weight function, such that

(1) $-k<f(e)<k$, for each edge e;

(2) the net outflow from each vertex in G is zero.

(refer to [Y] for properties of integer flow). The support of a k-flow f is the set of all edges with non-zero weights. A positive k-flow is a k-flow such that $f(e)>0$ for every edge e of G and a nowhere-zero k-flow is a k-flow such that $f(e)\neq 0$ for every edge e of G. (Note, we allow the negative weight in the general definition of k-flow. The existences of positive k-flow and nowhere-zero k-flow are equivalent for any graph since a positive k-flow can be obtained by changing the signs of negative weights and reversing the directions of edges with negative weights in a nowhere-zero k-flow.) For a simple graph $G=(V,E)$, an orientation D of G can be considered as a mapping $D\colon V(G) \times V(G) \rightarrow \{-1,0,1\}$ such that $D(uv) = 0$ if $uv \notin E(G)$, $D(uv)=1$ (or -1) if $uv \in E(G)$ and the direction of uv is from u to v (or from v to u, respectively). It is clear that $D(uv)=-D(vu)$. For the sake of convenience, we write $D(uv)f(uv)=Df(uv)$ if uv is an edge of a simple graph G with a k-flow (D, f). Thus the definition of a k-flow (D, f) of a simple graph G can be written as

(1) $-k < f(e) < k$ for each edge e of G,

(2) $\sum_{u\in N(v)} Df(vu) = 0$ for each vertex v of G.

Let $C=v_1v_2...v_rv_1$ be a cycle of a simple graph G which has a k-flow (D, f). A 2-flow $(D, t_{D,C})$ with support E(C) is defined as $Dt_{D,C}(v_iv_{i+1})=Dt_{D,C}(v_rv_1)=1$ for i=1,...,r-1 and $t_{D,C}(uv)=0$ if $uv\notin E(C)$. That is, $t_{D,C}(v_iv_{i+1})=1$ (or -1) if the direction of the edge v_iv_{i+1} in D is from v_i to v_{i+1} (or from v_{i+1} to v_i, respectively). An even subgraph H of G is a union of edge-disjoint cycles $\{C_i=v_1...v_{r_i}: i=1,...,s\}$. A 2-flow $(D, t_{D,H})$ of G with support E(H) can be defined as the sum of (D,t_{D,C_i}). That is, $t_{D,H}=\sum_{i=1}^{s} t_{D,C_i}$.

The minimum cycle covering problem (abbreviated to MCC) is to find a family F of cycles (or even subgraphs) in a graph G such that each edge of G is contained in some cycle(s) (or even subgraph(s)) of F and the total length of cycles in F is minimum. This problem has been studied extensively in recent years (see [AT], [AZ], [BJJ], [F1], [F2], [FP], [GF], [IR], [J1], [S1], [S2], etc.). *To find a* solution of MCC for a general graph might be an NP-hard problem[2]. Efforts have been made by mathematicians to estimate upper bounds of the solutions of MCC for general graphs or certain families of graphs (see [AT], [BJJ], [F1], [F2], [FP], etc.) and to determine the equivalence of MCC and the Chinese postman problem (see [EJ] or [BM], abbreviated to CPP), which is a polynomially solvable problem, for certain families of graphs (see [AZ], [GF], [IR], [S1]). One of the main results in this paper shows that the optimum solution of CPP is equivalent to the solution of MCC for any graph admitting a nowhere-zero 4-flow. Another main result in this paper verifies a conjecture by Fan ([F1]), which gives an upper bound of the solution of MCC for graphs admitting a nowhere-zero 3-flows.

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2 The author cannot find any reference on the complexity of MCC. He would appreciate if anyone could provide information about this problem.

A weight $w: E(G) \to \{1,2\}$ is called a (1,2)-eulerian weight of the graph G if the total weight of each edge-cut is even. Let (D, f) be an integer flow of G. The parity weight w_f of f is a weight function on E(G) such that

$$w_f(e) = \begin{cases} 1 & \text{if } f(e) \text{ is odd} \\ 2 & \text{if } f(e) \text{ is even} \end{cases}$$

for all e in E(G). It is easy to verify that w_f is a (1,2)-eulerian weight of G since (D,f) is an integer flow. If a graph G with a (1,2)-eulerian weight w has a family of even subgraphs such that each edge e of G with weight w(e) is contained in exactly w(e) even subgraphs of the family, then this graph G is said to be faithfully covered by the family of even subgraphs with respect to the weight w. The concept of faithful cycle covering problem was introduced and studied in [S1], [S2], [AZ] and [AGZ].

Denote

$$E_{f=even} = \{e \in E(G): f(e) \text{ is even}\}$$

and

$$E_{f=odd} = \{e \in E(G): f(e) \text{ is odd}\},$$

where f is the weight function of an integer flow (D,f) or an eulerian weight of G. It is trivial that $E_{f=odd}$ is an even subgraph of G.

The following theorem might be known by some experts in this field. Since the proof of the first part of the following theorem has not been seen in any publication, an outline of the proof will be included in this paper. (Note: The second part of the theorem was contained in the proof of Theorem 4.1 in [F1]).

THEOREM 1

Let G be a graph having a nowhere-zero 4-flow (D,f) and w be a (1,2)-eulerian weight of G. Then G can be faithfully covered

(i) by three even subgraphs with respect to w;

(ii) by two even subgraphs with respect to the parity weight of f.

PROOF.

Let $F_0 = E_{w=odd}$ and $F_1 = E_{f=odd}$. It is easy to verify that $(D, f_2) = (D, \frac{1}{2}(f+t_{D,F_1}))$ and $(D, f_3) = (D, \frac{1}{2}(f-t_{D,F_1}))$ are 3-flows of G. Let $F_i = E_{f_i=odd}$ for i=2,3. Then G can be faithfully covered by $\{F_2, F_3\}$ with respect to the parity weight of f and can be faithfully covered by $\{F_0 \Delta F_1, F_0 \Delta F_2, F_0 \Delta F_3\}$ with respect to the weight w.

###

COROLLARY 2

Let G be a graph having a nowhere-zero 4-flow (D, f) and w be a (1,2)-eulerian weight of G. Then G can be covered by

(i) three even subgraphs with total size

$$|E(G)| + |E_{w=even}|,$$

and (ii) two even subgraphs with total size

$$|E(G)| + |E_{f=even}|.$$

This is an immediate corollary of Theorem 1. (The second part of the Corollary was contained in the proof of Theorem 4.1 in [F1]).

Let T be a shortest closed tour of G (the optimum solution of CPP) with total length L(T), and let $\mathcal{F}$ be a family of cycles covering E(G) with the minimum total length $L(\mathcal{F}) = \sum_{c \in \mathcal{F}} L(C)$ (the solution of MCC). The optimum solution T of CPP is equivalent to the solution $\mathcal{F}$ of MCC if $L(T) = L(\mathcal{F})$. It was pointed out in various articles (eg. [IR], [GF]) that $L(T) \leq L(\mathcal{F})$ for any graph. It has also been proved that CPP and MCC are equivalent for 2-edge-connected planar graphs

([GF]), for 2-edge-connected graphs without subdivision of the Peterson graph ([AZ] and [AGZ]).

The following proposition shows a relation between the faithful cycle covering problem and the equivalence of CPP and MCC.

PROPOSITION 3

If a graph G is faithfully coverable with respect to *any* eulerian (1,2)-weight, then the optimum solution of the Chinese postman problem and the solution of MCC are equivalent.

PROOF.

We only need to show that the optimum solution of the Chinese postman problem is not less than the solution of MCC. Let T be an optimum solution of the Chinese postman problem for G. Define a weight w_T: $E(G) \rightarrow \{1,2\}$ by $w_T(e) = h$ if L passes through the edge e h times. Since G is faithfully coverable with respect to *any* eulerian (1,2)-weight, there is a family of cycles faithfully covering E(G) with respect to the weight w_T and therefore the proof of the proposition is completed.

###

Here an immediate application of Theorem 1 and Proposition 3 shows that

COROLLARY 4[3]

If a graph G has a nowhere-zero 4-flow, then the optimum solution of the Chinese postman problem and the solution of MCC are equivalent.

3. Corollary 4 was also independently proved by B. Jackson recently (see [JB])

Let $\mathcal{Z}_1$ be the set of all 2-edge-connected graphs containing no subdivision of the Peterson graph, $\mathcal{Z}_2$ be the the set of all 2-edge-connected graphs which have nowhere-zero 4-flows, $\mathcal{Z}_3$ be the set of all 2-edge-connected graphs which are faithfully coverable with respect to *any* eulerian (1,2)-weight and $\mathcal{Z}_4$ be the set of all graphs for which the optimum solution of the Chinese postman problem is equivalent to the the solution of the minimum cycle covering problem.

Problem 5[4]

(i) Is $\mathcal{Z}_3 = \mathcal{Z}_4$?

(ii) Is $\mathcal{Z}_2 = \mathcal{Z}_3$?

(iii) Is $\mathcal{Z}_2 = \mathcal{Z}_4$?

It was proved that $\mathcal{Z}_1 \subseteq \mathcal{Z}_3$ (see [AGZ]), $\mathcal{Z}_2 \subseteq \mathcal{Z}_3$ (Theorem 1), $\mathcal{Z}_3 \subseteq \mathcal{Z}_4$ (Proposition 3). The problems (ii) and (iii) are certainly stronger than Tutte's conjecture that $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$ (the 4-flow conjecture, see [T] or [Y]).

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[4] . Problems (ii) and (iii) were also independently proposed by H. Lai (personal communication)

It was proved (see [BJJ] or [F1]) that every graph $G=(V,E)$ admitting a nowhere-zero 4-flow can be covered by two even subgraphs of total size no more than $\frac{4}{3}|E(G)|$ and $|V(G)|+|E(G)|-1$. The following theorem was conjectured by Fan in [F1].

THEOREM 6.

If a simple graph $G=(V,E)$ has a nowhere-zero 3-flow, then G can be covered by two even subgraphs with the total size at most $|E(G)| + |V(G)| -3$.

By Corollary 2 and the equivalence of nowhere-zero k-flow and positive k-flow, we only need to prove the following lemma,

LEMMA 7

If a simple graph $G=(V,E)$ has a positive 3-flow, then G has a positive 4-flow (D, f) such that $|E_{f=even}| \leq |V(G)| -3$.

Both Theorem 6 and Lemma 7 are best possible since the complete bipartite graph $K_{3,3k}$ (k is a positive integer) has a positive 3-flow but the total size of a cycle cover is at least $|E|+|V|-3$ (see [F1]). Also the condition of the existence of a nowhere-zero 3-flow cannot be weakened since K_4 is a graph having a positive 4-flow for which the total size of a cycle cover is at least $|E|+|V|-2$.

Let $\{V', V''\}$ be a partition of $V(G)$. The edge-cut between V' and V'' is denoted by $E(V', V'')$. If D is an orientation of $E(G)$, let

$$E_{D+}(V', V'') = \{uv\in E(G): u\in V' \text{ and } v\in V'', D(uv)=1\}$$

and $$E_{D-}(V', V'') = \{uv\in E(G): u\in V' \text{ and } v\in V'', D(uv)=-1\}$$

PROOF OF LEMMA 7

I. Let (D, f_o) be a positive 3-flow on G with an orientation D. Since f_o is positive, it is well-known that

(1) $$\frac{1}{2}|E_{D^+}(T',T'')| \leq |E_{D^-}(T',T'')| \leq 2|E_{D^+}(T',T'')|$$

II. Let (D, f) be a positive 4-flow of G such that the set $E_{f=even}$ is as small as possible (Note that the positive 3-flow (D, f_0) is a positive 4-flow of G with respect to the orientation D). We claim that $E_{f=even}$ is a forest of G. If not, let $C' = v_1 \ldots v_r v_1$ be a cycle in $E_{f=even}$. Then $(D, f') = (D, f+t_{D,C'})$ is a positive 4-flow of G for which the set $E_{f'=even} = E_{f=even} \backslash E(C')$ is smaller than $E_{f=even}$. This contradicts the definition of (D, f).

III. Let G_2 be a subgraph of G such that $E(G_2) = E_{f=even}$ and $V(G_2) = V(G)$. By II, the subgraph G_2 is a spanning forest of G. If G_2 has at least three components, then $|E(G_2)| = |E_{f=even}| \leq |V(G)|-3$ and the lemma is proved. Thus we assume that G_2 has at most two components T_1 and T_2.

IV. We claim that G_2 has exactly two components T_1, T_2 and $E(G) \backslash E_{f=even}$ is exactly the edge cut between T_1 and T_2. That is, each edge of G with an odd weight must join a pair vertices of two different components. Suppose not, let x_0x_1 be an edge of $E(G) \backslash E_{f=even}$ such that both x_0 and x_1 are in the same component T of G_2. Note that $Df(x_0x_1) \in \{1, -1\ 3, -3\}$. If $Df(x_0x_1) \in \{1, -3\}$ (or $\in \{-1, 3\}$). Let $C''=x_0x_1 \ldots x_r x_0$ be the cycle in $T_1 \cup \{x_0x_1\}$. Then $(D, f'') = (D, f+t_{D,C''})$ (or $= (D, f-t_{D,C''})$, respectively) is a positive 4-flow of G which has a smaller

$$E_{f''=even} = [E_{f=even} \backslash E(C'')] \cup \{x_0x_1\}$$

since G is a simple graph. This contradicts the choice of the 4-flow (D, f).

V. Note that $Df(uv) \in \{\pm 1, \pm 3\}$ for each edge $uv \in E(T_1, T_2) = E(G) \backslash E_{f=even}$. Denote

$L^+ = \{uv \in E(G): u \in T_1,\ v \in T_2,\ Df(uv) = 1 \text{ or } -3\}$

and $L^- = \{uv \in E(G): u \in T_1,\ v \in T_2,\ Df(uv) = 3 \text{ or } -1\}$.

We claim that neither L^+ nor L^- is empty. Suppose that L^- is empty. Thus, for each edge uv, $f(uv)=1$ if $uv \in E_{D+}(T_1,T_2)$ or $f(uv)=3$ if $uv \in E_{D-}(T_1,T_2)$. Since (D, f) is a flow of G, we must have that $|E_{D+}(T_1,T_2)| = 3|E_{D-}(T_1,T_2)|$. This contradicts the inequality (1).

VI. Since the case of the graphs of order at most three is trivial, we will assume that the graph G contains at least four vertices. Therefore we assume that T_1 has at least two vertices and u be an endvertex of the tree T_1. Since the edge incident with u in T_1 has weight 2 and (D, f) is a 4-flow, u must be incident with at least two edges of odd weights which are in $E(T_1,T_2)$. Since G is a simple graph, T_2 must have at least two vertices adjacent to u. Hence both T_1 and T_2 are non-trivial trees.

VII. Since neither T_1 nor T_2 is a single vertex, the diameters of T_1 and T_2 are at least one. Assume that the diameter of T_1 is not less than the diameter of T_2. It is very easy to see that if the diameter of T_1 is one then G must be the complete graph K_4 which cannot have a positive 3-flow. Thus the diameter of T_1 is at least two.

VIII. We claim that there is no cycle C^* in G such that the length of C^* is at least five and $|C^* \cap L^+| = |C^* \cap L^-| = 1$. If not, let $C^*=uv...yx...u$ be such a cycle with $uv \in L^+$ and $xy \in L^-$. The flow $(D, f^*) = (D, f+t_{D,C^*})$ is a positive 4-flow of G. But the set

$$E_{f^*=even} = [E_{f=even} \backslash E(C^*)] \cup \{uv, xy\}$$

is smaller than $E_{f=even}$ and this contradicts the choice of (D,f).

IX. Let u be an endvertex of the tree T_1. We claim that u must be incident with at least two edges of L^+ or two edges of L^-. If not, then

$$\sum_{v \in N(u)} Df(uv) \neq 0,$$

and this contradicts the definition of (D, f).

X. Without loss of generality, we assume that an endvertex u of T_1 is incident with at least two edges ux and uy of L^+. Let w be a vertex of T_1 incident with an edge wz of L^-. We claim that the distance between w and u is at most one in T_1. Assume, without loss of generality, that $x \neq z$. There is a cycle $C^\circ = ux...zw...u$ in G such that $|C^\circ \cap L^+| = |C^\circ \cap L^-| = 1$. If the distance between w and u is at least two in T_1, then the length of C° is at least five and contradicts the conclusion of VIII.

XI. Since the distance between any pair of endvertices of T_1 is at least two and one endvertex u is incident with two edges of L^+, an endvertex of T_1 other than u cannot be incident with any edge of L^- (by X). By X again, u cannot be incident with any edge of L^- because any endvertex of T_1 other than u is incident with at least two edges of L^+.

XII. Since L^- is not empty by V, let w be a vertex of T_1 incident with some edge of L^-. By XI and X, the distance between w and every endvertex of T_1 is one in T_1. Thus w is the only vertex of T_1 incident with some edge of L^- and T_1 is a star with w as its center.

XIII. Let u_1, u_2 be two endvertices of T_1 and w be the center of T_1. Since u_i is incident with at least two edges of L^+ for i=1 and 2, let u_1x_1, u_2x_2 be two disjoint edges of L^+. That is, $x_1 \neq x_2$. Let wy be an edge of L^-. We claim that the distance between y and x_i in T_2 is at most one for i=1 and 2. If not, let the distance between x_1 and y be at least two in T_2. Then there is a cycle $C\# = u_1x_1 \ldots ywu_1$ in G of length at least five with $|C\# \cap L^+| = |C\# \cap L^-| = 1$. This contradicts the conclusion of VIII. Thus the subgraph of T_2 induced by $\{x_1, x_2, y\}$ is a path of length one or two with x_1 and x_2 as its endvertices.

XIV. Let $C_i = u_ix_iywu_i$ if $x_i \neq y$ or $C_i = u_ix_iwu_i$ if $x_i = y$ for i=1 and 2. Then the flow $(D, f_y) = (D, f+t_{D,C_1}+t_{D,C_2})$ is a positive 4-flow with a smaller

$$E_{f_y=\mathrm{even}} = \{E_{f=\mathrm{even}} \setminus [E(C_1) \cup E(C_2)]\} \cup \{u_1x_1, u_2x_2\}.$$

This contradicts the choice of (D, f) and completes the proof of the Lemma.

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